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Integrable and superintegrable systems with spin in three-dimensional Euclidean space

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Abstract

A systematic search for superintegrable quantum Hamiltonians describing the interaction between two particles with spins 0 and $\frac{1}{2}$ is performed. We restrict to integrals of motion that are first-order (matrix) polynomials in the components of linear momentum. Several such systems are found and for one nontrivial example we show how superintegrability leads to exact solvability: we obtain exact (nonperturbative) bound-state energy formulas and exact expressions for the wave functions in terms of products of Laguerre and Jacobi polynomials.

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1. Introduction

The purpose of this research program is to perform a systematic study of integrability and superintegrability in the interaction of two particles with spin. Specifically in this paper we consider a system of two nonrelativistic particles, one with spin $s = \frac{1}{2}$ (e.g. a nucleon) and the other with spin $s = 0$ (e.g. a pion), moving in the three-dimensional Euclidean space E_3 .

The Pauli–Schrödinger equation in this case will have the form

$$H\Psi = \left(-\frac{\hbar^2}{2}\Delta + V_0(\vec{x}) + \frac{1}{2}\{V_1(\vec{x}), (\vec{\sigma}, \vec{L})\} \right) \Psi = E\Psi, \quad (1.1)$$

where the $V_1(\vec{x})$ term represents the spin–orbital interaction. We use the notation

$$p_1 = -i\hbar\partial_x, \quad p_2 = -i\hbar\partial_y, \quad p_3 = -i\hbar\partial_z, \quad (1.2)$$

$$L_1 = i\hbar(z\partial_y - y\partial_z), \quad L_2 = i\hbar(x\partial_z - z\partial_x), \quad L_3 = i\hbar(y\partial_x - x\partial_y), \quad (1.3)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.4)$$

for the linear momentum, angular momentum and Pauli matrices, respectively. The curly bracket in (1.1) denotes an anticommutator.

For spinless particles, the Hamiltonian

$$H = -\frac{\hbar^2}{2}\Delta + V_0(\vec{x}) \quad (1.5)$$

is a scalar operator, whereas H in (1.1) is a 2×2 matrix operator and $\Psi(\vec{x})$ is a two-component spinor.

In the spinless case (1.5), the Hamiltonian is integrable if there exists a pair of commuting integrals of motion X_1, X_2 that are well-defined quantum mechanical operators, such that H, X_1 and X_2 are algebraically independent. If further algebraically independent integrals Y_i exist, the system is superintegrable. The best-known superintegrable systems in E_3 are the hydrogen atom and the harmonic oscillator. Each of them is maximally superintegrable with $2n - 1 = 5$ independent integrals, generating an $o(4)$ and an $su(3)$ algebra, respectively [1, 2].

A systematic search for quantum and classical superintegrable scalar potentials in (1.5) with integrals that are first- and second-order polynomials in the momenta was performed some time ago [3–5]. First-order integrals correspond to geometrical symmetries of the potential, and second-order ones are directly related to the separation of variables in the Schrödinger equation or the Hamilton–Jacobi equation in the classical case [3, 6–8].

First- and second-order integrals of motion are rather easy to find for Hamiltonians of the type (1.5) in Euclidean space. The situation with third- and higher-order integrals is much more difficult [9–11].

If a vector potential term corresponding e.g. to a magnetic field is added, the problem becomes much more difficult and the existence of second-order integrals no longer implies the separation of variables [12, 13].

Case (1.1) with a spin–orbital interaction turns out to be quite rich and rather difficult to treat systematically. In a previous article, we have considered the same problem in E_2 [14]. Here we concentrate on the Hamiltonian (1.1) in E_3 but restrict to first-order integrals. Thus, we search for integrals of motion of the form

$$X = (A_0 + \vec{A} \cdot \vec{\sigma})p_1 + (B_0 + \vec{B} \cdot \vec{\sigma})p_2 + (C_0 + \vec{C} \cdot \vec{\sigma})p_3 + \phi_0 + \vec{\phi} \cdot \vec{\sigma} - \frac{i\hbar}{2}\{(A_0 + \vec{A} \cdot \vec{\sigma})_x + (B_0 + \vec{B} \cdot \vec{\sigma})_y + (C_0 + \vec{C} \cdot \vec{\sigma})_z\}, \quad (1.6)$$

where A_0, B_0, C_0, ϕ_0 and A_i, B_i, C_i, ϕ_i ($i = 1, 2, 3$) are all scalar functions of \vec{x} .

In section 2, we show that a spin–orbital interaction of the form

$$V_1 = \frac{\hbar}{r^2} \quad (1.7)$$

can be induced by a gauge transformation from a purely scalar potential $V_0(\vec{x})$ (in particular from $V_0 = 0$). In section 3, we derive and discuss the determining equations for the existence of first-order integrals. In section 4, we restrict to rotationally invariant potentials $V_0(r)$ and $V_1(r)$ and classify the integrals of motion into $O(3)$ multiplets. Solutions of the determining equations are obtained in section 5. Superintegrable potentials are discussed in section 6. In section 7, we solve the Pauli–Schrödinger equation for one superintegrable system explicitly and exactly. Finally, the conclusions and outlook are given in section 8.

2. Spin–orbital interaction induced by a gauge transformation

2.1. Derivation

In this subsection, we show that a spin–orbit term could be gauge induced from a scalar Hamiltonian (1.5) by a gauge transformation. The transformation matrix must be an element of $U(2)$:

$$U = e^{i\beta_4} \begin{pmatrix} e^{i\beta_1} \cos(\beta_3) & e^{i\beta_2} \sin(\beta_3) \\ -e^{-i\beta_2} \sin(\beta_3) & e^{-i\beta_1} \cos(\beta_3) \end{pmatrix}, \quad (2.1)$$

where β_j ($j = 1, 2, 3, 4$) are the real functions of (x, y, z) . It is seen that in order to generate a spin-orbit term, we need to have

$$\hbar^2 U^\dagger (\vec{\nabla} U) \cdot \vec{\nabla} = \Gamma(\vec{\sigma}, \vec{L}), \quad (2.2)$$

where Γ is an arbitrary real scalar function of (x, y, z) . Equation (2.2) implies 12 first-order partial differential equations for β_j and Γ , three of which are $\beta_{4,k} = 0$, $k = 1, 2, 3$. Hence, without loss of generality we choose $\beta_4 = 0$ and then write the remaining nine equations as

$$\cos^2(\beta_3)\beta_{1,z} - \sin^2(\beta_3)\beta_{2,z} = 0, \quad (2.3)$$

$$\cos^2(\beta_3)\beta_{1,x} - \sin^2(\beta_3)\beta_{2,x} = 0, \quad (2.4)$$

$$\hbar(\cos^2(\beta_3)\beta_{1,y} - \sin^2(\beta_3)\beta_{2,y}) = -x\Gamma, \quad (2.5)$$

$$\hbar(\cos(\beta_2 - \beta_1)\beta_{3,x} - \frac{1}{2}\sin(\beta_2 - \beta_1)\sin(2\beta_3)(\beta_2 + \beta_1)_x) = -z\Gamma, \quad (2.6)$$

$$\hbar(\cos(\beta_2 - \beta_1)\beta_{3,z} - \frac{1}{2}\sin(\beta_2 - \beta_1)\sin(2\beta_3)(\beta_2 + \beta_1)_z) = x\Gamma, \quad (2.7)$$

$$\hbar(\sin(\beta_2 - \beta_1)\beta_{3,y} + \frac{1}{2}\cos(\beta_2 - \beta_1)\sin(2\beta_3)(\beta_2 + \beta_1)_y) = z\Gamma, \quad (2.8)$$

$$\hbar(\sin(\beta_2 - \beta_1)\beta_{3,z} + \frac{1}{2}\cos(\beta_2 - \beta_1)\sin(2\beta_3)(\beta_2 + \beta_1)_z) = -y\Gamma, \quad (2.9)$$

$$\cos(\beta_2 - \beta_1)\beta_{3,y} - \frac{1}{2}\sin(\beta_2 - \beta_1)\sin(2\beta_3)(\beta_2 + \beta_1)_y = 0, \quad (2.10)$$

$$\sin(\beta_2 - \beta_1)\beta_{3,x} + \frac{1}{2}\cos(\beta_2 - \beta_1)\sin(2\beta_3)(\beta_2 + \beta_1)_x = 0. \quad (2.11)$$

From equations (2.6)–(2.11), we obtain

$$\begin{aligned} \beta_j &= \beta_j(\xi, \eta), \quad j = 1, 2, 3, \quad \text{where } \xi = \frac{x}{z}, \quad \eta = \frac{y}{z}, \\ \cos(\beta_2 - \beta_1) &= -\frac{\beta_{3,\xi}}{\sqrt{\beta_{3,\xi}^2 + \beta_{3,\eta}^2}}, \quad \sin(\beta_2 - \beta_1) = \frac{\beta_{3,\eta}}{\sqrt{\beta_{3,\xi}^2 + \beta_{3,\eta}^2}}, \\ (\beta_2 + \beta_1)_\xi &= \frac{2\beta_{3,\eta}}{\sin(2\beta_3)}, \quad (\beta_2 + \beta_1)_\eta = -\frac{2\beta_{3,\xi}}{\sin(2\beta_3)}, \end{aligned} \quad (2.12)$$

and

$$\Gamma = \hbar \frac{\sqrt{\beta_{3,\xi}^2 + \beta_{3,\eta}^2}}{z^2}. \quad (2.13)$$

Introducing $\beta_j(\xi, \eta)$ into equations (2.3)–(2.5) and using the compatibility condition for the mixed partial derivatives of $(\beta_2 + \beta_1)$, we obtain the following three partial differential

equations:

$$\begin{aligned} \beta_{3,\xi\xi} + \beta_{3,\eta\eta} &= 2(\beta_{3,\xi}^2 + \beta_{3,\eta}^2) \cot(2\beta_3), \\ \beta_{3,\eta}\beta_{3,\xi\eta} - \beta_{3,\xi}\beta_{3,\eta\eta} &= 2(\beta_{3,\xi}^2 + \beta_{3,\eta}^2)(\xi\sqrt{\beta_{3,\xi}^2 + \beta_{3,\eta}^2} - \cot(2\beta_3)\beta_{3,\xi}), \\ -\beta_{3,\eta}\beta_{3,\xi\xi} + \beta_{3,\xi}\beta_{3,\xi\eta} &= 2(\beta_{3,\xi}^2 + \beta_{3,\eta}^2)(\eta\sqrt{\beta_{3,\xi}^2 + \beta_{3,\eta}^2} - \cot(2\beta_3)\beta_{3,\eta}), \end{aligned} \quad (2.14)$$

which could be solved for the highest-order derivatives of β_3 (i.e. $\beta_{3,\xi\xi}$, $\beta_{3,\xi\eta}$ and $\beta_{3,\eta\eta}$). Then, the compatibility conditions of these give

$$\sqrt{\beta_{3,\xi}^2 + \beta_{3,\eta}^2} = -\frac{1}{\xi^2 + \eta^2 + 1}, \quad (2.15)$$

which implies that $\Gamma = -\frac{\hbar}{r^2}$. Hence, we conclude that $V_1 = \frac{\hbar}{r^2}$ is gauge induced and it is the only potential which could be generated from a scalar Hamiltonian by a gauge transformation.

The explicit form of the gauge transformation U is found as

$$\begin{aligned} \beta_1 &= \varphi + c_1, & \beta_2 &= c_2, & \beta_3 &= -\theta + c_3, \\ \beta_4 &= 0, & 0 &\leq \theta \leq \pi, & 0 &\leq \varphi < 2\pi, \end{aligned} \quad (2.16)$$

where c_1 , c_2 and c_3 are the following constants:

$$c_1 = c_2 \pm \pi, \quad c_3 = \pm \frac{\pi}{2} \quad (2.17)$$

and θ , φ are the spherical coordinates.

With this transformation matrix, the transformed Hamiltonian is found to be

$$\tilde{H} = U^{-1} \left(-\frac{\hbar^2}{2} \Delta + V_0(\vec{x}) \right) U = -\frac{\hbar^2}{2} \Delta + V_0(\vec{x}) + \frac{\hbar^2}{r^2} + \frac{\hbar}{r^2} \vec{\sigma} \cdot \vec{L}. \quad (2.18)$$

2.2. Integrals for $V_0 = V_0(r)$ and $V_1 = \frac{\hbar}{r^2}$

The potential $V_1 = \frac{\hbar}{r^2}$ is gauge induced from a Hamiltonian of the form (1.5) (though each term is multiplied by a 2×2 identity matrix). Hence the integrals for this case are just the gauge transforms of the integrals of motion of this Hamiltonian (i.e. L_j and σ_j). They can be written as

$$J_i = L_i + \frac{\hbar}{2} \sigma_i, \quad S_i = -\frac{\hbar}{2} \sigma_i + \hbar \frac{x_i}{r^2} (\vec{r}, \vec{\sigma}), \quad (2.19)$$

and satisfy the following commutation relations:

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k, \quad [S_i, S_j] = i\hbar \epsilon_{ijk} S_k, \quad [J_i, S_j] = i\hbar \epsilon_{ijk} S_k. \quad (2.20)$$

The Lie algebra \mathcal{L} is isomorphic to a direct sum of the algebra $o(3)$ with itself:

$$\mathcal{L} \sim o(3) \oplus o(3) = \{\vec{J} - \vec{S}\} \oplus \{\vec{S}\}. \quad (2.21)$$

2.3. Integrals for $V_0 = \frac{\hbar^2}{r^2}$ and $V_1 = \frac{\hbar}{r^2}$

Since these potentials are gauge induced from a free Hamiltonian, the integrals are just the gauge transforms of L_j , p_j and σ_j , which can be written as

$$J_i = L_i + \frac{\hbar}{2} \sigma_i, \quad \Pi_i = p_i - \frac{\hbar}{r^2} \epsilon_{ikl} x_k \sigma_l, \quad S_i = -\frac{\hbar}{2} \sigma_i + \hbar \frac{x_i}{r^2} (\vec{r}, \vec{\sigma}). \quad (2.22)$$

They satisfy the following commutation relations:

$$\begin{aligned} [J_i - S_i, S_j] &= 0, & [\Pi_i, S_j] &= 0, & [\Pi_i, \Pi_j] &= 0, \\ [J_i - S_i, J_j - S_j] &= i\hbar\epsilon_{ijk}(J_k - S_k), & [J_i - S_i, \Pi_j] &= i\hbar\epsilon_{ijk}\Pi_k. \end{aligned} \tag{2.23}$$

Hence the nine-dimensional Lie algebra \mathcal{L} is isomorphic to a direct sum of the Euclidean Lie algebra $e(3)$ with the algebra $o(3)$:

$$\mathcal{L} \sim e(3) \oplus o(3) = \{\vec{J} - \vec{S}, \vec{\Pi}\} \oplus \{\vec{S}\}. \tag{2.24}$$

3. Determining equations for an integral of motion

3.1. Derivation

In this subsection, we give the full set of determining equations obtained from the commutativity condition $[H, X] = 0$, where H is the Hamiltonian given in (1.1) and X is the most general first-order integral of motion given in (1.6). This commutator has second-, first- and zero-order terms in the momenta. By setting the coefficients of different powers of the momenta equal to zero in each entry of this 2×2 matrix, we obtain the following determining equations. Since the Planck constant \hbar enters into the determining equations in a nontrivial way, we keep it throughout the whole set of determining equations. However, after giving the determining equations we set $\hbar = 1$ for simplicity.

3.1.1. Determining equations obtained from the second-order terms. From the diagonal elements, it is immediately found that A_0, B_0 and C_0 are linear functions and are expressed for any potentials V_0 and V_1 as

$$A_0 = b_1 - a_3y + a_2z, \quad B_0 = b_2 + a_3x - a_1z, \quad C_0 = b_3 - a_2x + a_1y, \tag{3.1}$$

where a_i and b_i ($i = 1, 2, 3$) are real constants. After introducing (3.1) into the rest of the coefficients of the second-order terms and separating the imaginary and real parts of the coefficients obtained from the off-diagonal elements, we are left with an overdetermined system of 18 partial differential equations for A_i, B_i, C_i ($i = 1, 2, 3$) and V_1 . These are

$$\begin{aligned} 2yA_1V_1 + \hbar A_{3,x} &= 0, & 2yA_1V_1 + \hbar A_{2,x} &= 0, & 2xB_2V_1 + \hbar B_{1,y} &= 0, \\ 2zB_2V_1 + \hbar B_{3,y} &= 0, & 2xC_3V_1 + \hbar C_{1,z} &= 0, & 2yC_3V_1 + \hbar C_{2,z} &= 0, \\ 2V_1(yA_2 + zA_3) - \hbar A_{1,x} &= 0, & 2V_1(xB_1 + zB_3) - \hbar B_{2,y} &= 0, \\ 2V_1(xC_1 + yC_2) - \hbar C_{3,z} &= 0, & 2zV_1(A_2 + B_1) + \hbar A_{3,y} + \hbar B_{3,x} &= 0, \\ 2yV_1(A_3 + C_1) + \hbar A_{2,z} + \hbar C_{2,x} &= 0, & 2xV_1(B_3 + C_2) + \hbar B_{1,z} + \hbar C_{1,y} &= 0, \\ 2V_1(xA_1 + yA_2 - zC_1) - \hbar A_{3,z} - \hbar C_{3,x} &= 0, \\ 2V_1(xB_1 + yB_2 - zC_2) - \hbar B_{3,z} - \hbar C_{3,y} &= 0, \\ 2V_1(xA_2 - yB_2 - zB_3) + \hbar A_{1,y} + \hbar B_{1,x} &= 0, \\ 2V_1(xA_1 + zA_3 - yB_1) - \hbar A_{2,y} - \hbar B_{2,x} &= 0, \\ 2V_1(xA_3 - yC_2 - zC_3) + \hbar A_{1,z} + \hbar C_{1,x} &= 0, \\ 2V_1(yB_3 - xC_1 - zC_3) + \hbar B_{2,z} + \hbar C_{2,y} &= 0. \end{aligned} \tag{3.2}$$

3.1.2. *Determining equations obtained from the first-order terms.* After introducing (3.1) and separating the real and imaginary parts, we have the following 12 partial differential equations:

$$\begin{aligned}
 V_1(\hbar(b_1 - a_3y) + 2y\phi_3) + \hbar(x(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) + \phi_{2,z}) &= 0, \\
 V_1(\hbar(b_1 + a_2z) - 2z\phi_2) + \hbar(x(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) - \phi_{3,y}) &= 0, \\
 V_1(\hbar(b_2 - a_1z) + 2z\phi_1) + \hbar(y(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) + \phi_{3,x}) &= 0, \\
 V_1(\hbar(b_2 + a_3x) - 2x\phi_3) + \hbar(y(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) - \phi_{1,z}) &= 0, \\
 V_1(\hbar(b_3 - a_2x) + 2x\phi_2) + \hbar(z(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) + \phi_{1,y}) &= 0, \\
 V_1(\hbar(b_3 + a_1y) - 2y\phi_1) + \hbar(z(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) - \phi_{2,x}) &= 0, \\
 V_1(\hbar(a_2y + a_3z) - 2y\phi_2 - 2z\phi_3) + \hbar\phi_{1,x} &= 0, \\
 V_1(\hbar(a_1x + a_3z) - 2x\phi_1 - 2z\phi_3) + \hbar\phi_{2,y} &= 0, \\
 V_1(\hbar(a_1x + a_2y) - 2x\phi_1 - 2y\phi_2) + \hbar\phi_{3,z} &= 0,
 \end{aligned} \tag{3.3}$$

$$\begin{aligned}
 \phi_{0,x} &= V_1((yA_{3,x} - xA_{3,y}) + (xA_{2,z} - zA_{2,x}) + (zA_{1,y} - yA_{1,z}) + (C_2 - B_3)) \\
 &\quad + V_{1,x}(zA_2 - yA_3) + V_{1,y}(zB_2 - yB_3) + V_{1,z}(zC_2 - yC_3), \\
 \phi_{0,y} &= V_1((yB_{3,x} - xB_{3,y}) + (xB_{2,z} - zB_{2,x}) + (zB_{1,y} - yB_{1,z}) + (A_3 - C_1)) \\
 &\quad + V_{1,x}(xA_3 - zA_1) + V_{1,y}(xB_3 - zB_1) + V_{1,z}(xC_3 - zC_1), \\
 \phi_{0,z} &= V_1((yC_{3,x} - xC_{3,y}) + (xC_{2,z} - zC_{2,x}) + (zC_{1,y} - yC_{1,z}) + (B_1 - A_2)) \\
 &\quad + V_{1,x}(yA_1 - xA_2) + V_{1,y}(yB_1 - xB_2) + V_{1,z}(yC_1 - xC_2),
 \end{aligned} \tag{3.4}$$

where A_0, B_0 and C_0 are given in (3.1). There are also nine other second-order partial differential equations for A_i, B_i, C_i and V_1 , obtained from the coefficients of the first-order terms. However, these are differential consequences of (3.2) so we do not present them here.

3.1.3. *Determining equations obtained from the zero-order terms.* Setting the coefficients of the zero-order terms in each entry of the commutation relation equal to zero and separating the real and imaginary parts, we have the following four partial differential equations:

$$\begin{aligned}
 A_0V_{0,x} + B_0V_{0,y} + C_0V_{0,z} + V_1(x(\phi_{2,z} - \phi_{3,y}) + y(\phi_{3,x} - \phi_{1,z}) + z(\phi_{1,y} - \phi_{2,x})) &= 0, \\
 \hbar\{V_1(A_{1,z} + B_{2,z} - C_{2,y} - C_{1,x}) + V_{1,x}((xA_{1,z} - zA_{1,x}) + (yA_{2,z} - zA_{2,y})) \\
 &\quad + V_{1,y}((xB_{1,z} - zB_{1,x}) + (yB_{2,z} - zB_{2,y})) \\
 &\quad + V_{1,z}((xC_{1,z} - zC_{1,x}) + (yC_{2,z} - zC_{2,y}))\} \\
 &\quad + 2(A_3V_{0,x} + B_3V_{0,y} + C_3V_{0,z}) + 2V_1(y\phi_{0,x} - x\phi_{0,y}) = 0, \\
 \hbar\{V_1(B_{2,x} + C_{3,x} - A_{3,z} - A_{2,y}) + V_{1,x}((zA_{3,x} - xA_{3,z}) + (yA_{2,x} - xA_{2,y})) \\
 &\quad + V_{1,y}((zB_{3,x} - xB_{3,z}) + (yB_{2,x} - xB_{2,y})) \\
 &\quad + V_{1,z}((zC_{3,x} - xC_{3,z}) + (yC_{2,x} - xC_{2,y}))\} \\
 &\quad + 2(A_1V_{0,x} + B_1V_{0,y} + C_1V_{0,z}) + 2V_1(z\phi_{0,y} - y\phi_{0,z}) = 0, \\
 \hbar\{V_1(A_{1,y} + C_{3,y} - B_{3,z} - B_{1,x}) + V_{1,x}((zA_{3,y} - yA_{3,z}) + (xA_{1,y} - yA_{1,x})) \\
 &\quad + V_{1,y}((zB_{3,y} - yB_{3,z}) + (xB_{1,y} - yB_{1,x})) \\
 &\quad + V_{1,z}((zC_{3,y} - yC_{3,z}) + (xC_{1,y} - yC_{1,x}))\} \\
 &\quad + 2(A_2V_{0,x} + B_2V_{0,y} + C_2V_{0,z}) + 2V_1(x\phi_{0,z} - z\phi_{0,x}) = 0.
 \end{aligned} \tag{3.5}$$

In general, the partial differential equations in (3.5) involve second- and third-order derivatives of A_i, B_i and C_i ($i = 1, 2, 3$); however, using (3.2) these terms can be eliminated.

3.2. Discussion of solution in the general case

In general, the solution of the determining equations (3.2)–(3.5) for the 15 unknowns ϕ_0, V_0, V_1 and A_i, B_i, C_i, ϕ_i ($i = 1, 2, 3$) turns out to be a difficult problem. However, it is seen that the determining equations (3.3) do not involve ϕ_0, A_i, B_i, C_i ($i = 1, 2, 3$) and V_0 and hence could be analyzed separately.

In order to determine the unknown functions V_1 and ϕ_i , we express the first-order derivatives of ϕ_i s from (3.3) and require the compatibility of the mixed partial derivatives. This requirement gives us another nine equations for ϕ_i s and first-order derivatives of them. Now, if we introduce the first-order derivatives of ϕ_i s from (3.3) into this system, we get a system of algebraic equations for ϕ_i s ($i = 1, 2, 3$). This system of algebraic equations can be written in the following way:

$$M \cdot \Phi = R, \tag{3.6}$$

where M is a 9×3 matrix and Φ and R are 3×1 and 9×1 vectors, respectively. The matrix M can be written as

$$M = \begin{pmatrix} 0 & 2\delta_1 & 2z\delta_3 \\ 0 & -2x\delta_2 & 2x\delta_3 \\ 0 & 2y\delta_2 & 2\delta_5 \\ -2\delta_1 & 0 & -2z\delta_4 \\ 2x\delta_2 & 0 & 2\delta_6 \\ -2y\delta_2 & 0 & 2y\delta_4 \\ -2z\delta_3 & 2z\delta_4 & 0 \\ -2x\delta_3 & -2\delta_6 & 0 \\ -2\delta_5 & -2y\delta_4 & 0 \end{pmatrix}, \tag{3.7}$$

where δ_i ($i = 1, \dots, 6$) are defined as follows:

$$\begin{aligned} \delta_1 &= 2V_1 - 2z^2V_1^2 + yV_{1,y} + xV_{1,x}, & \delta_2 &= 2zV_1^2 + V_{1,z}, \\ \delta_3 &= 2yV_1^2 + V_{1,y}, & \delta_4 &= 2xV_1^2 + V_{1,x}, \\ \delta_5 &= 2V_1 - 2y^2V_1^2 + zV_{1,z} + xV_{1,x}, & \delta_6 &= 2V_1 - 2x^2V_1^2 + yV_{1,y} + zV_{1,z}. \end{aligned} \tag{3.8}$$

The vector Φ is $(\phi_1, \phi_2, \phi_3)^T$ and the entries of the vector R are given as follows:

$$\begin{aligned} R_{11} &= 2V_1(a_2 - 2xz(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z})) + a_2(xV_{1,x} + yV_{1,y} + zV_{1,z}) \\ &\quad - 2(b_3x + zA_0)V_1^2 - b_3V_{1,x} - z(A_0V_{1,xx} + B_0V_{1,xy} + C_0V_{1,xz}), \\ R_{21} &= y(A_0V_{1,xy} + B_0V_{1,yy} + C_0V_{1,yz}) + z(A_0V_{1,xz} + B_0V_{1,yz} + C_0V_{1,zz}) \\ &\quad - 2x(b_1 + A_0)V_1^2 + (3 - 4x^2V_1)(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) - b_1V_{1,x}, \\ R_{31} &= 2V_1(a_3 + 2xy(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z})) + a_3(xV_{1,x} + yV_{1,y} + zV_{1,z}) \\ &\quad + 2(b_2x + yA_0)V_1^2 + b_2V_{1,x} + y(A_0V_{1,xx} + B_0V_{1,xy} + C_0V_{1,xz}), \\ R_{41} &= -2V_1(a_1 + 2yz(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z})) - a_1(xV_{1,x} + yV_{1,y} + zV_{1,z}) \\ &\quad - 2(b_3y + zB_0)V_1^2 - b_3V_{1,y} - z(A_0V_{1,xy} + B_0V_{1,yy} + C_0V_{1,yz}), \\ R_{51} &= 2V_1(a_3 - 2xy(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z})) + a_3(xV_{1,x} + yV_{1,y} + zV_{1,z}) \\ &\quad - 2(b_1y + xB_0)V_1^2 - b_1V_{1,y} - x(A_0V_{1,xy} + B_0V_{1,yy} + C_0V_{1,yz}), \\ R_{61} &= -x(A_0V_{1,xx} + B_0V_{1,xy} + C_0V_{1,xz}) - z(A_0V_{1,xz} + B_0V_{1,yz} + C_0V_{1,zz}) \\ &\quad + 2y(b_2 + B_0)V_1^2 - (3 - 4y^2V_1)(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) + b_2V_{1,y}, \end{aligned} \tag{3.9}$$

$$\begin{aligned}
 R_{71} &= x(A_0V_{1,xx} + B_0V_{1,xy} + C_0V_{1,xz}) + y(A_0V_{1,xy} + B_0V_{1,yy} + C_0V_{1,yz}) \\
 &\quad - 2z(b_3 + C_0)V_1^2 + (3 - 4z^2V_1)(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z}) - b_3V_{1,z}, \\
 R_{81} &= -2V_1(a_2 + 2xz(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z})) - a_2(xV_{1,x} + yV_{1,y} + zV_{1,z}) \\
 &\quad - 2(b_1z + xC_0)V_1^2 - b_1V_{1,z} - x(A_0V_{1,xz} + B_0V_{1,yz} + C_0V_{1,zz}), \\
 R_{91} &= 2V_1(-a_1 + 2yz(A_0V_{1,x} + B_0V_{1,y} + C_0V_{1,z})) - a_1(xV_{1,x} + yV_{1,y} + zV_{1,z}) \\
 &\quad + 2(b_2z + yC_0)V_1^2 + b_2V_{1,z} + y(A_0V_{1,xz} + B_0V_{1,yz} + C_0V_{1,zz}),
 \end{aligned}$$

where A_0 , B_0 and C_0 are given in (3.1).

In general, the rank of matrix M is 3 and the rank of the extended matrix of the system (3.6) is 4. Thus, we have to analyze under which conditions we can equate these two ranks. The rank of matrix M can be 0, 2 or 3. It can be 2 if and only if V_1 satisfies the following differential equation:

$$2V_1 + xV_{1,x} + yV_{1,y} + zV_{1,z} = 0, \tag{3.10}$$

which implies

$$V_1 = \frac{F(\xi, \eta)}{r^2}, \quad F \neq 1, \tag{3.11}$$

where $\xi = \frac{x}{z}$ and $\eta = \frac{y}{z}$. It can be 0 if and only if $V_1 = \frac{1}{r^2}$. Hence, the rank of matrix M and the rank of the extended matrix can be equal if we have the following.

- (i) For rank= 0, $V_1 = \frac{1}{r^2}$ with $a_i \neq 0$ and $b_i \neq 0, i = 1, 2, 3$.
- (ii) For rank= 3, $V_1 = V_1(r)$ and $b_i = 0$ for all i and $a_i \neq 0, i = 1, 2, 3$.

For the former case, the system (3.6) has the following solution:

$$\begin{aligned}
 \phi_1 &= \frac{a_1}{2} + \frac{(x^2 - y^2 - z^2)\alpha_1 + 2xy\alpha_2 + 2xz\alpha_3 - 2b_2z + 2b_3y}{2(x^2 + y^2 + z^2)}, \\
 \phi_2 &= \frac{a_2}{2} + \frac{(y^2 - x^2 - z^2)\alpha_2 + 2xy\alpha_1 + 2yz\alpha_3 + 2b_1z - 2b_3x}{2(x^2 + y^2 + z^2)}, \\
 \phi_3 &= \frac{a_3}{2} + \frac{(z^2 - x^2 - y^2)\alpha_3 + 2xz\alpha_1 + 2yz\alpha_2 - 2b_1y + 2b_2x}{2(x^2 + y^2 + z^2)},
 \end{aligned} \tag{3.12}$$

where α_1, α_2 and α_3 are real constants and for the latter it has

$$\phi_1 = \frac{a_1}{2}, \quad \phi_2 = \frac{a_2}{2}, \quad \phi_3 = \frac{a_3}{2}. \tag{3.13}$$

Indeed for the potential $V_1 = \frac{1}{r^2}$, the whole set of determining equations (3.2)–(3.5) can be solved and we obtain a nine-dimensional Lie algebra \mathcal{L} given in (2.24). For $V_1 = V_1(r), V_0 = V_0(r)$ we obtain the well-known result that H commutes with the total angular momentum $\vec{J} = \vec{L} + \frac{1}{2}\vec{\sigma}$.

In the rest of the paper, we restrict to spherically symmetric potentials $V_1 = V_1(r), V_0 = V_0(r)$ and have a rotationally invariant Hamiltonian. However, the integrals of motion would transform under the action of rotations $O(3)$. Instead of solving the whole set of determining equations, we analyze the problem by classifying the integrals of motion into irreducible $O(3)$ multiplets.

4. Classification of integrals of motion into $O(3)$ multiplets

Let us assume that the Hamiltonian is

$$H = -\frac{1}{2}\Delta + V_0(r) + V_1(r)\vec{\sigma} \cdot \vec{L}, \tag{4.1}$$

and that X of (1.6) is an integral of motion. Rotations in $E(3)$ leave the Hamiltonian invariant but can transform X into new invariants. We can hence decompose the space of integrals of motion into subspaces transforming under irreducible representations of $O(3)$. We shall also require that the subspaces have definite behavior under the parity operator.

At our disposal are two vectors \vec{x} and \vec{p} and one pseudovector $\vec{\sigma}$. The integrals we are considering can involve at most first-order powers of \vec{p} and $\vec{\sigma}$ but arbitrary powers of \vec{x} .

General formulas for the decomposition of the representation $[D(j)]^n$ of $O(3)$ into irreducible components are given by Murnaghan [15]. Since we are interested only in $j = 1$, we shall proceed *ab initio* rather than specialize his results.

We shall construct scalars, pseudoscalars, vectors, axial vectors and symmetric two-component tensors and pseudotensors in the space:

$$\{ \{\vec{x}\}^n \times \vec{p} \times \vec{\sigma} \}. \tag{4.2}$$

The quantities \vec{x} , \vec{p} and $\vec{\sigma}$ allow us to define six independent ‘directions’ in the direct product of the Euclidean space and the spin one, namely

$$\{ \vec{x}, \vec{p}, \vec{L} = \vec{x} \wedge \vec{p}, \vec{\sigma}, \vec{\sigma} \wedge \vec{x}, \vec{\sigma} \wedge \vec{p} \}, \tag{4.3}$$

and any $O(3)$ tensor can be expressed in terms of these. The positive integer n in (4.2) is arbitrary and any scalar in \vec{x} space will be written as $f(r)$, where f is an arbitrary function of $r = \sqrt{x^2 + y^2 + z^2}$. Since $\vec{\sigma}$ and \vec{p} figure at most linearly, we can form exactly three independent scalars and three pseudoscalars out of the quantities (4.3).

Scalars:

$$S_1 = 1, \quad S_2 = (\vec{x}, \vec{p}), \quad S_3 = (\vec{\sigma}, \vec{L}). \tag{4.4}$$

Pseudoscalars:

$$P_1 = (\vec{\sigma}, \vec{x}), \quad P_2 = (\vec{\sigma}, \vec{p}), \quad P_3 = (\vec{x}, \vec{p})(\vec{\sigma}, \vec{x}). \tag{4.5}$$

The independent vectors and axial vectors are as follows.

Vectors:

$$\begin{aligned} \vec{V}_1 &= \vec{x}, & \vec{V}_2 &= \vec{p}, & \vec{V}_3 &= (\vec{x}, \vec{p})\vec{x}, & \vec{V}_4 &= (\vec{\sigma}, \vec{L})\vec{x}, \\ \vec{V}_5 &= (\vec{x}, \vec{p})(\vec{\sigma} \wedge \vec{x}), & \vec{V}_6 &= \vec{\sigma} \wedge \vec{p}, & \vec{V}_7 &= \vec{\sigma} \wedge \vec{x}, & \vec{V}_8 &= (\vec{\sigma}, \vec{x})\vec{L}. \end{aligned} \tag{4.6}$$

Axial vectors:

$$\begin{aligned} \vec{A}_1 &= \vec{L}, & \vec{A}_2 &= \vec{\sigma}, & \vec{A}_3 &= (\vec{x}, \vec{p})\vec{\sigma}, & \vec{A}_4 &= (\vec{\sigma}, \vec{p})\vec{x}, \\ \vec{A}_5 &= (\vec{x}, \vec{\sigma})\vec{x}, & \vec{A}_6 &= (\vec{x}, \vec{\sigma})\vec{p}, & \vec{A}_7 &= (\vec{x}, \vec{p})(\vec{\sigma}, \vec{x})\vec{x}. \end{aligned} \tag{4.7}$$

Similarly, we can form ten independent two-component symmetric tensors and nine symmetric pseudotensors.

Tensors:

$$\begin{aligned} T_1^{ik} &= x^i x^k, & T_2^{ik} &= (\vec{x}, \vec{p})x^i x^k, & T_3^{ik} &= (\vec{\sigma}, \vec{L})x^i x^k, & T_4^{ik} &= x^i p^k + x^k p^i, \\ T_5^{ik} &= (\vec{\sigma}, \vec{x})(x^i L^k + x^k L^i), & T_6^{ik} &= x^i (\vec{\sigma} \wedge \vec{x})^k + x^k (\vec{\sigma} \wedge \vec{x})^i, \\ T_7^{ik} &= (\vec{x}, \vec{p})(x^i (\vec{\sigma} \wedge \vec{x})^k + x^k (\vec{\sigma} \wedge \vec{x})^i), & T_8^{ik} &= x^i (\vec{\sigma} \wedge \vec{p})^k + x^k (\vec{\sigma} \wedge \vec{p})^i, \\ T_9^{ik} &= p^i (\vec{\sigma} \wedge \vec{x})^k + p^k (\vec{\sigma} \wedge \vec{x})^i, & T_{10}^{ik} &= L^i \sigma^k + L^k \sigma^i. \end{aligned} \tag{4.8}$$

Pseudotensors:

$$\begin{aligned} Y_1^{ik} &= (\vec{\sigma}, \vec{p})x^i x^k, & Y_2^{ik} &= (\vec{\sigma}, \vec{x})x^i x^k, & Y_3^{ik} &= (\vec{x}, \vec{p})(\vec{\sigma}, \vec{x})x^i x^k, \\ Y_4^{ik} &= (\vec{\sigma}, \vec{x})(x^i p^k + x^k p^i), & Y_5^{ik} &= x^i L^k + x^k L^i, \\ Y_6^{ik} &= x^i \sigma^k + x^k \sigma^i, & Y_7^{ik} &= (\vec{x}, \vec{p})(x^i \sigma^k + x^k \sigma^i), \\ Y_8^{ik} &= p^i \sigma^k + p^k \sigma^i, & Y_9^{ik} &= L^i (\vec{\sigma} \wedge \vec{x})^k + L^k (\vec{\sigma} \wedge \vec{x})^i. \end{aligned} \tag{4.9}$$

An arbitrary function $f(r)$ is also a scalar and each of the quantities in (4.4)–(4.9) can be multiplied by $f(r)$ without changing its properties under rotations or reflections.

All the tensors and pseudotensors should be considered to be traceless since their traces appear separately as scalars, or pseudoscalars. For simplicity of notation, we do not subtract the trace explicitly.

5. Solution of commutativity equations for $V_0 = V_0(r)$, $V_1 = V_1(r)$

In this section, we separately take the linear combinations of all the scalars, pseudoscalars, vectors, axial vectors and two-component tensors and pseudotensors. For simplicity of notation we just write the bare linear combinations; however, in the analysis of the commutation relations we use the full symmetric form of those, which could be found in the appendix.

5.1. Scalars

Let us take a linear combination of the scalars given in (4.4):

$$X_S = \sum_{j=1}^3 f_j(r) S_j. \tag{5.1}$$

It is immediately seen that in order to satisfy the commutativity equation $[H, X_S] = 0$ we must have

$$f_1 = c_1, \quad f_2 = 0, \quad f_3 = c_2, \tag{5.2}$$

where c_1 and c_2 are real constants. The corresponding integrals S_1 and S_3 are trivial.

5.2. Pseudoscalars

As an integral of motion, we take a linear combination of the pseudoscalars given in (4.5):

$$X_P = \sum_{j=1}^3 f_j(r) P_j, \tag{5.3}$$

and require $[H, X_P] = 0$. The determining equations, obtained by equating the coefficients of the second-order terms to zero in the commutativity equation, become

$$f_3' = -2rf_3V_1, \quad f_3 = -2f_2V_1, \tag{5.4}$$

$$2rV_1^2(2r^2V_1 - 3) - V_1' = 0. \tag{5.5}$$

Depending on the solutions of the compatibility equation for V_1 (5.5), we have several cases. The solution of (5.5) is given by

$$V_1 = \frac{1}{2r^2} \left(1 + \frac{\alpha}{\sqrt{1 + \beta r^2}} \right), \tag{5.6}$$

where $\alpha^2 = 1$. Note that $V_1 = \frac{1}{r^2}$ and $V_1 = \frac{1}{2r^2}$ are the special solutions of (5.5) with $(\alpha, \beta) = (1, 0)$ and $(1, \infty)$, respectively. The case $V_1 = \frac{1}{r^2}$ induced by a gauge transformation has already been considered.

Case I: $V_1 = \frac{1}{2r^2}$

For this type of potential, (5.4) implies that

$$f_2 = -c_1r \quad \text{and} \quad f_3 = \frac{c_1}{r}, \tag{5.7}$$

and the first-order equations give

$$f_1 = \frac{c_2}{r}. \tag{5.8}$$

Then zero-order equations are satisfied for any $V_0(r)$ and we have an integral of motion X_P for the above values of f_i .

Case II: $V_1 = \frac{1}{2r^2} \left(1 + \frac{\alpha}{\sqrt{1+\beta r^2}}\right)$, $0 < \beta < \infty$, $\alpha^2 = 1$

For this type of potential, (5.4) implies that

$$f_2 = -\frac{c_1}{\beta} \sqrt{1 + \beta r^2} \quad \text{and} \quad f_3 = \frac{c_1}{-\alpha + \sqrt{1 + \beta r^2}}, \tag{5.9}$$

and the first-order equations give $f_1 = 0$. Then V_0 is determined from the zero-order equations to be $V_0 = V_1$ and we have an integral of motion X_P for the above values of f_i .

5.3. Vectors

Let us now take the linear combination of the vectors given in (4.6):

$$\vec{X}_V = \sum_{j=1}^8 f_j(r) \vec{V}_j, \tag{5.10}$$

and require $[H, \vec{X}_V] = 0$. The second-order terms give

$$f_2 = c_1, \quad f_3 = 0, \quad f_4 + f_5 = 0, \tag{5.11}$$

$$r^2 f_4' = f_6' - 2r f_4, \tag{5.12}$$

$$f_4 - 2f_6 V_1 + f_8(1 - 2r^2 V_1) = 0, \tag{5.13}$$

$$f_8' = -2r V_1 (f_8 + f_4) - f_4'. \tag{5.14}$$

Equation (5.12) implies

$$f_4 = \frac{c_2 + f_6}{r^2}. \tag{5.15}$$

Introducing (5.15) into (5.13) and solving for f_8 , we obtain

$$f_8 = -\frac{c_2 + f_6(1 - 2r^2 V_1)}{r^2(1 - 2r^2 V_1)}. \tag{5.16}$$

Finally, if we introduce f_4 and f_8 into (5.14), we get a compatibility condition for V_1 which is exactly the same as (5.5). Thus, we have the following cases.

Case I: $V_1 = \frac{1}{2r^2}$

For this type of potential, the commutativity equation $[H, \vec{X}_V] = 0$ implies $f_2 = f_3 = f_7 = 0$, $f_4 = -f_5 = \frac{f_6}{r^2}$, $f_1 = \frac{c_3}{2r}$ and $f_8 = \frac{c_3}{r} - \frac{f_6}{r^2}$. Thus we have an integral of motion \vec{X}_V for these values of f_i .

Case II: $V_1 = \frac{1}{2r^2} \left(1 + \frac{\alpha}{\sqrt{1+\beta r^2}}\right)$

For this type of potential, the commutativity equation $[H, \vec{X}_V] = 0$ implies $f_1 = f_2 = f_3 = f_7 = 0$, $-f_4 = f_5 = f_8 = -\frac{f_6}{r^2}$. However, if we introduce these values of f_i into the integral of motion \vec{X}_V in (5.10), it identically vanishes. Hence, we do not have an integral of motion.

Case III: $V_1 = V_1(r)$ and $f_4 = 0$, $f_6 = 0$ and $f_8 = 0$

For this case, equations (5.12)–(5.14) are all satisfied and we have

$$f_2 = c_1, \quad f_3 = 0, \quad f_4 = 0, \quad f_5 = 0, \quad f_6 = 0, \quad f_8 = 0. \quad (5.17)$$

Then first-order terms determine f_1 , f_7 as

$$f_1 = 0, \quad f_7' + 2rf_7V_1 = 0, \quad 2rf_7V_1 + c_1V_1' = 0, \quad (5.18)$$

and we conclude that we either have $V_1 = \frac{1}{r^2}$ or we do not have any integral of motion (i.e. $f_j = 0$ for all $j = 1, \dots, 8$).

5.4. Axial vectors

Let us now take the linear combination of the axial vectors given in (4.7):

$$\vec{X}_A = \sum_{j=1}^7 f_j(r) \vec{A}_j, \quad (5.19)$$

and require $[H, \vec{X}_A] = 0$. The second-order terms give

$$f_1 = c_1, \quad f_3 = 0, \quad (5.20)$$

$$f_4' = 4rf_4V_1(1 - r^2V_1), \quad (5.21)$$

$$f_6' = 2r(f_4 - f_6)V_1, \quad f_4 = -f_6(1 - 2r^2V_1), \quad (5.22)$$

$$f_7' + 2rf_7V_1 = 0, \quad f_7 = -2f_4V_1. \quad (5.23)$$

Equations (5.21) and (5.22) imply

$$f_6(2V_1(3 - 6r^2V_1 + 4r^4V_1^2) + rV_1') = 0, \quad (5.24)$$

where as (5.21) and (5.23) imply

$$f_4(2rV_1^2(3 - 2r^2V_1) + V_1') = 0. \quad (5.25)$$

The only common solutions of (5.24) and (5.25) are $V_1 = \frac{1}{r^2}$ and $V_1 = \frac{1}{2r^2}$. Thus, other than $V_1 = \frac{1}{r^2}$, we have two cases.

Case I: $V_1 = \frac{1}{2r^2}$

For this type of potential, (5.21)–(5.23) give

$$f_4 = 0, \quad f_6 = \frac{c_2}{r^2}, \quad f_7 = 0, \quad (5.26)$$

and then the first-order terms imply

$$f_2 = \frac{c_1}{2}, \quad f_5 = 0, \quad c_2 = 0. \quad (5.27)$$

The zero-order terms are satisfied for arbitrary V_0 . Thus, the only integral of motion is $\vec{X}_A = \vec{L} + \frac{1}{2}\vec{\sigma}$, the total angular momentum (an integral for any $V_1(r)$ and $V_0(r)$).

Case II: $V_1 = V_1(r)$ and $f_4 = 0$, $f_6 = 0$, $f_7 = 0$

For $f_4 = 0$, $f_6 = 0$, $f_7 = 0$, (5.21)–(5.23) are all satisfied for arbitrary $V_1(r)$. The first-order terms give

$$f_2 = c_2, \quad f_5 = (c_1 - 2c_2)V_1, \quad (5.28)$$

$$f_5' + 2rf_5V_1 = 0. \quad (5.29)$$

However, (5.29) together with (5.28) implies

$$(c_1 - 2c_2)(V_1' + 2rV_1^2) = 0. \tag{5.30}$$

Thus, we have two more subcases:

- (i) $c_1 = 2c_2$. We have $f_5 = 0$ and $f_2 = \frac{c_1}{2}$ and the rest of the determining equations are satisfied for arbitrary $V_1(r)$ and $V_0(r)$. Hence the only integral of motion is the total angular momentum.
- (ii) $V_1' + 2rV_1^2 = 0$. We have $V_1 = \frac{1}{r^2 - \alpha}$. Introducing this V_1 together with (5.28) into the determining equations, we obtain $(c_1 - 2c_2)\alpha = 0$. Hence we conclude that we either have $\alpha = 0$ (i.e. $V_1 = \frac{1}{r^2}$) or $c_1 = 2c_2$, both of which have already been investigated.

5.5. Tensors

Let us now take the linear combination of the tensors given in (4.8):

$$X_T^{ik} = \sum_{j=1}^{10} f_j(r) T_j^{ik}, \tag{5.31}$$

and require $[H, X_T^{ik}] = 0$. The second-order terms give

$$f_2 = 0, \quad f_4 = 0, \quad f_3' = -2f_7', \tag{5.32}$$

$$f_3 + f_5 + f_7 + 2(f_{10} - f_9)V_1 = 0, \quad f_3 + 2f_5 + 4r^2(f_{10} - f_9)V_1^2 = 0, \tag{5.33}$$

$$f_7 - f_5 + 2(r^2f_5 + f_8 + f_{10})V_1 = 0, \tag{5.34}$$

$$f_7' - 2r(f_5 - f_7)V_1 - f_5' = 0, \tag{5.35}$$

$$f_8 + f_9 + r^2(f_5 + 2(f_{10} - f_9)V_1) = 0, \tag{5.36}$$

$$f_{10}' - f_9' + 4r(f_{10} - f_9)V_1(1 - r^2V_1) = 0, \tag{5.37}$$

$$(f_{10} - f_9)(2V_1(3 - 6r^2V_1 + 4r^4V_1^2) + rV_1') = 0. \tag{5.38}$$

Equation (5.38) implies either $f_{10} = f_9$ or a compatibility condition for V_1 which has the following solution:

$$V_1 = \frac{1}{2r^2} \pm \frac{1}{\sqrt{4r^4 + \alpha}}. \tag{5.39}$$

Note that the potentials $V_1 = \frac{1}{r^2}$ and $V_1 = \frac{1}{2r^2}$ are also solutions which correspond to limiting values of α (i.e. $\alpha = 0$ and $\alpha = \infty$). Thus, we have the following cases.

Case I: $f_{10} = f_9$

Equations (5.33)–(5.36) give

$$f_3 = -2f_5, \quad f_7 = f_5, \quad f_8 = -(f_{10} + r^2f_5), \tag{5.40}$$

for arbitrary $V_1(r)$ and the first-order terms imply

$$f_1 = 0, \quad f_6 = 0. \tag{5.41}$$

Then the zero-order terms are satisfied for any $V_1(r)$ and $V_0(r)$. However, if we introduce the above values of f_i into the integral of motion X_T^{12} in (5.31), it identically vanishes. Hence, we do not have any integral of motion for this case.

Case II: $V_1 = \frac{1}{2r^2}$

For this type of potential, (5.33)–(5.37) imply

$$f_{10} - f_9 = \frac{c_1}{r}, \quad f_3 + 2f_5 = -\frac{c_1}{r^3}, \quad r^2f_5 + f_8 + f_{10} = 0, \quad f_7 = f_5. \tag{5.42}$$

However, then (5.32) implies $c_1 = 0$ and we are back in case I.

Case III: $V_1 = \frac{1}{2r^2} \pm \frac{1}{\sqrt{4r^4 + \alpha}}$

For this type of potential, (5.37) implies

$$f_{10} = \frac{c_1}{r} (4r^4 + \alpha)^{\frac{1}{4}} + f_9, \tag{5.43}$$

and (5.34) together with (5.36) gives

$$f_7 = \pm \frac{4c_1 r}{(4r^4 + \alpha)^{\frac{1}{4}}} V_1 + f_5. \tag{5.44}$$

However, if we introduce (5.44) into (5.35), we see that we must have

$$c_1 \alpha = 0. \tag{5.45}$$

Hence, we either have $c_1 = 0$ or $\alpha = 0$, both of which have already been investigated.

5.6. Pseudotensors

Let us now take the linear combination of the pseudotensors given in (4.9):

$$X_Y^{ik} = \sum_{j=1}^9 f_j(r) Y_j^{ik}, \tag{5.46}$$

and require $[H, X_Y^{ik}] = 0$. The second-order terms give

$$f_3 = -2(f_1 + 2f_9)V_1, \quad f_4 = -2f_8 V_1 + (1 - 2r^2 V_1) f_9, \quad f_5 = 0, \quad f_7 = f_9, \tag{5.47}$$

$$f_1 = 2(2f_8 V_1(1 - r^2 V_1) - f_9(1 - 2r^2 V_1(1 - r^2 V_1))), \quad f_8' = -r(2f_9 + r f_9'), \tag{5.48}$$

$$(f_8 + r^2 f_9)(2r V_1^2(3 - 2r^2 V_1) - V_1') + f_9' = 0, \tag{5.49}$$

$$(f_8 + r^2 f_9)(2r V_1^2(3 - 4r^2 V_1 + 2r^4 V_1^2) - (1 - 2r^2 V_1)V_1') = 0, \tag{5.50}$$

$$(f_8 + r^2 f_9)(2r V_1^2(3 - 4r^2 V_1) + (6r^2 V_1(1 - r^2 V_1) - 1)V_1') = 0. \tag{5.51}$$

Equations (5.50) and (5.51) imply that we either have $f_8 + r^2 f_9 = 0$ or $V_1 = \frac{1}{r^2}$. Hence, other than $V_1 = \frac{1}{r^2}$, we have the following case.

Case I: $f_8 + r^2 f_9 = 0$

Equations (5.47)–(5.49) give

$$f_1 = 2c_1, \quad f_3 = 0, \quad f_4 = f_7 = f_9 = -c_1, \quad f_8 = r^2 c_1. \tag{5.52}$$

Then the first-order terms give

$$f_2 = 0, \quad f_6 = 0. \tag{5.53}$$

However, upon introducing the above values of f_j ($j = 1, \dots, 9$) into the integral of motion X_Y^{12} in (5.46), it vanishes identically and we do not have an integral of motion for this case.

6. Discussion of results

For any spherically symmetric potentials $V_0(r)$ and $V_1(r)$, it is well known that $J_i = L_i + \frac{1}{2}\sigma_i$ and $(\vec{\sigma}, \vec{L})$ are integrals of motion, where \vec{J} is an axial vector and $(\vec{\sigma}, \vec{L})$ is a scalar.

Additional first-order integrals exist only in four special cases. Two of them are treated in sections 2.2 and 2.3 and the spin-orbital term $V_1 = \frac{1}{r^2}$ is gauge induced.

For $V_0 = V_1 = \frac{1}{r^2}$, we obtain the vector $\vec{\Pi}$ and the axial vectors \vec{J} and \vec{S} of (2.22). In addition we obtain the following.

Pseudoscalar:

$$X_P = -\frac{1}{2}(\vec{\sigma}, \vec{p}) + \frac{1}{r^2}(\vec{\sigma}, \vec{x})(\vec{x}, \vec{p}) - \frac{i}{r^2}(\vec{\sigma}, \vec{x}). \quad (6.1)$$

Vector:

$$\vec{V} = \frac{2\vec{x}}{r^2} - (\vec{\sigma} \wedge \vec{p}) + \frac{2}{r^2}(\vec{\sigma}, \vec{x})(\vec{x} \wedge \vec{p}) - \frac{i(\vec{x} \wedge \vec{\sigma})}{r^2}. \quad (6.2)$$

Axial vector:

$$\vec{A} = -\frac{1}{2}\vec{x}(\vec{\sigma}, \vec{p}) + \frac{1}{2}\vec{\sigma} - \frac{1}{2}(\vec{\sigma}, \vec{x})\vec{p} + \frac{\vec{x}}{r^2}(\vec{\sigma}, \vec{x})(\vec{x}, \vec{p}) - i\frac{3\vec{x}}{2r^2}(\vec{\sigma}, \vec{x}). \quad (6.3)$$

However, they all lie in the enveloping algebra of the Lie algebra (2.24).

For $V_1 = \frac{1}{r^2}$ and $V_0 = V_0(r) \neq V_1$, we reobtain the algebra (2.21) and the gauge transforms of the axial vector $\vec{\sigma} \wedge \vec{L}$ and the tensor $\sigma^j L^k$ that are in the enveloping algebra of (2.21).

For $V_1 = \frac{1}{2r^2}$ and $V_0 = V_0(r)$ we have the following.

Pseudoscalars:

$$X_P^1 = \frac{(\vec{\sigma}, \vec{x})}{r}, \quad (6.4)$$

$$X_P^2 = -r(\vec{\sigma}, \vec{p}) + \frac{1}{r}(\vec{\sigma}, \vec{x})(\vec{x}, \vec{p}) - \frac{i}{r}(\vec{\sigma}, \vec{x}). \quad (6.5)$$

Vector:

$$\vec{X}_V = \frac{1}{2r}\vec{x} + \frac{1}{r}(\vec{\sigma}, \vec{x})\vec{L} - \frac{i}{2r}(\vec{x} \wedge \vec{\sigma}). \quad (6.6)$$

For $V_0 = V_1 = \frac{1}{2r^2} \left(1 + \frac{\alpha}{\sqrt{1+\beta r^2}}\right)$, $\alpha^2 = 1$, we have the following pseudoscalar.

$$X_P = -\frac{1}{\beta}\sqrt{1+\beta r^2}(\vec{\sigma}, \vec{p}) + \frac{(\vec{\sigma}, \vec{x})}{-\alpha + \sqrt{1+\beta r^2}}((\vec{x}, \vec{p}) - i). \quad (6.7)$$

7. An example of an exact solution of the Pauli–Schrödinger equation

Usually the spin–orbital interaction term is treated perturbatively [16]. In the case of superintegrable systems we can obtain exact solutions. In this paper, we consider only one example: the potential $V_1(r) = \frac{1}{2r^2}$, $V_0 = V_0(r)$ and the integral of motion X_P^1 (6.4). The system of equations to solve is

$$H\Psi = E\Psi, \quad (7.1)$$

$$J^2\Psi = j(j+1)\Psi, \quad (7.2)$$

$$J_3\Psi = m\Psi, \quad (7.3)$$

$$X_P^1\Psi = \epsilon\Psi, \quad (7.4)$$

where $\Psi = \Psi_{njme}(r, \theta, \phi)$ is a two-component spinor.

Equation (7.3) implies

$$\Psi = \begin{pmatrix} f_1(r, \theta) e^{i(m-\frac{1}{2})\phi} \\ f_2(r, \theta) e^{i(m+\frac{1}{2})\phi} \end{pmatrix}. \tag{7.5}$$

Equation (7.4) relates f_1 and f_2 :

$$f_2(r, \theta) = \frac{\epsilon - \cos \theta}{\sin \theta} f_1(r, \theta), \quad \epsilon^2 = 1. \tag{7.6}$$

Equation (7.2) provides an equation for f_1 , namely

$$f_{1,\theta\theta} + \frac{\epsilon}{\sin \theta} f_{1,\theta} - \left\{ \frac{1}{\sin^2 \theta} \left(m - \frac{1}{2} \right) \left(m + \frac{1}{2} - \epsilon \cos \theta \right) - j(j+1) - \frac{1}{4} \right\} f_1 = 0, \tag{7.7}$$

with

$$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots, \quad m = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \pm j. \tag{7.8}$$

To solve (7.7) let us put

$$f_1 = R(r)F(\theta), \tag{7.9}$$

and obtain an equation for the angular part $F(\theta)$ from (7.7), namely

$$F_{\theta\theta} + \frac{\epsilon}{\sin \theta} F_{\theta} - \frac{1}{\sin^2 \theta} \left\{ m^2 - \frac{1}{4} - \epsilon \left(m - \frac{1}{2} \right) \cos \theta - \left(j(j+1) + \frac{1}{4} \right) \sin^2 \theta \right\} F = 0. \tag{7.10}$$

Equation (7.10) can be solved in terms of Jacobi polynomials. We get different expressions for $m > 0$ and $m < 0$.

For $m < 0$, we have

$$F(\theta) = (1 - z^2)^{\frac{1}{4} - \frac{m}{2}} P_{j+m}^{(\alpha, \beta)}(z), \tag{7.11}$$

$$\alpha = -m + \frac{\epsilon}{2}, \quad \beta = -m - \frac{\epsilon}{2}, \quad z = \cos \theta, \tag{7.12}$$

which is regular for $-1 \leq z \leq 1$ (for $m < 0$).

For $m > 0$, we have

$$F(\theta) = (1 - z)^{\frac{m}{2} - \frac{\epsilon}{2} + \frac{1}{4}} (1 + z)^{\frac{m}{2} + \frac{\epsilon}{2} + \frac{1}{4}} P_{j-m}^{(\alpha, \beta)}(z), \tag{7.13}$$

$$\alpha = m - \frac{\epsilon}{2}, \quad \beta = m + \frac{\epsilon}{2}, \quad z = \cos \theta. \tag{7.14}$$

Solution (7.13) is regular for $-1 \leq z \leq 1$ (for $m > 0$).

Finally, to obtain the radial part of the solution we put the results obtained so far into (7.1) and obtain the radial equation:

$$-\frac{1}{2} \left(R'' + \frac{2}{r} R' \right) + \left\{ V_0(r) + \left(j(j+1) - \frac{3}{4} \right) \frac{1}{2r^2} \right\} R = ER. \tag{7.15}$$

We shall solve (7.15) for the case when $V_0(r)$ is the Coulomb potential

$$V_0(r) = \frac{\mu}{r}, \quad \mu < 0. \tag{7.16}$$

The result is obtained in terms of Laguerre polynomials. We put

$$R(r) = e^{wr} r^p L(\sigma r), \tag{7.17}$$

and obtain

$$\sigma r L'' + 2 \left(p + 1 + \frac{w}{\sigma} \sigma r \right) L' + \left\{ \frac{1}{\sigma r} \left(p(p+1) - j(j+1) + \frac{3}{4} \right) + \frac{2}{\sigma} (w(p+1) - \mu) + \frac{w^2 + 2E}{\sigma^2} \sigma r \right\} L = 0. \tag{7.18}$$

This coincides with the equation for Laguerre polynomials $L = L_n^\alpha(\sigma r)$ if we put

$$p = -\frac{1}{2} + \sqrt{j^2 + j - \frac{1}{2}}, \quad w = -\sqrt{-2E}, \quad \sigma = 2\sqrt{-2E}, \tag{7.19}$$

$$\frac{2}{\sigma} (w(p+1) - \mu) = n, \quad \alpha = 2\sqrt{j^2 + j - \frac{1}{2}}.$$

From (7.18), we also find the bound state energies to be

$$E_{nj} = -\frac{\mu^2}{2\left(n + \frac{1}{2} + \sqrt{j^2 + j - \frac{1}{2}}\right)^2}. \tag{7.20}$$

We see that the energy depends on only two quantum numbers, n and j , whereas the wave function (7.5) depends on n, j, m and ϵ . The spin-orbital interaction removes the ‘dynamical’ or ‘accidental’ degeneracy with respect to the quantum number j . Superintegrability relates the two components of the spinor $\Psi_{njm\epsilon}$ and this made it possible to calculate the wave functions explicitly and exactly.

8. Conclusions and outlook

The main results of this paper can be summed up as a theorem.

Theorem 1. *The only first-order spherically symmetric superintegrable systems of type (1.1) in the Euclidean space E_3 are the following ones:*

$$1. \quad V_0 = \frac{\hbar^2}{r^2}, \quad V_1 = \frac{\hbar}{r^2}. \tag{8.1}$$

The integrals of motion are given in (2.22) and form the algebra (2.23), (2.24). The potentials (8.1) are induced from free motion by a gauge transformation.

$$2. \quad V_0 = V_0(r), \quad V_1 = \frac{\hbar}{r^2}, \tag{8.2}$$

where $V_0(r)$ is arbitrary. The integrals are given in (2.19) and form the algebra (2.20), (2.21). The spin-orbital term V_1 is induced by a gauge transformation.

$$3. \quad V_0 = V_0(r), \quad V_1 = \frac{\hbar}{2r^2}, \tag{8.3}$$

where $V_0(r)$ is arbitrary. The integrals are the two pseudoscalars (6.4), (6.5) and the vector (6.6).

$$4. \quad V_0 = \hbar V_1, \quad V_1 = \frac{\hbar}{2r^2} \left(1 + \frac{\alpha}{\sqrt{1 + \beta r^2}} \right), \quad \alpha^2 = 1. \tag{8.4}$$

The integral is the pseudoscalar (6.7).

In all cases the components of the total angular momentum $J_i = L_i + \frac{\hbar}{2}\sigma_i, i = 1, 2, 3$, are also integrals of motion.

In the case of scalar particles ($V_1(r) = 0$), first-order superintegrability does not exist. The best-known cases of second-order superintegrability are the Coulomb atom and the harmonic oscillator. In the first case, the additional (to angular momentum) integrals of motion form a vector (the Laplace–Runge–Lenz vector). In the second case, they form a two-valent tensor. In both cases, the integrals generate a non-Abelian Lie algebra and this leads to an additional degeneracy of the energy levels.

In the case considered in this paper the situation is different. First of all, first-order superintegrability does exist (see theorem 1). For cases 3 and 4 of the theorem, the additional pseudoscalar integrals commute with the total angular momentum. Hence it is possible to simultaneously diagonalize H, J^2, J_3 and the additional pseudoscalar integral X . In the example, considered in section 7, the energy depends on two quantum numbers n and j , the wave functions on n, j, m and $\epsilon = \pm 1$. We thus have the geometric degeneracy related to the operator J_3 and also a discrete degeneracy due to X .

Theorem 1 also provides examples of ‘pure quantum integrability’ [10, 11, 17]. The potentials V_1 and sometimes also the potentials V_0 disappear in the classical limit $\hbar \rightarrow 0$.

It has been conjectured [18] that all maximally superintegrable (scalar) systems are also exactly solvable. This means that their bound state energies can be calculated algebraically. Moreover, their wave functions can be expressed as polynomials in the appropriate variables, multiplied by some overall factor. Example (7.1)–(7.4) is superintegrable, but not maximally so. We have however shown that for $V_0 = \frac{\mu}{r}$ the system is exactly solvable.

The conjecture of [18] has been supported by many examples [18–20].

In a future article, we plan to study the potentials (8.3) and (8.4) in more detail, making other choices for V_0 in (8.3) and diagonalizing a more general operator $X_2 + \alpha X_1$ (see (6.4) and (6.5)).

Another project that is being pursued is that of second-order superintegrability. The Hamiltonian is the same as in (1.1); however, the integrals, in addition to the total angular momentum, are not of the form (1.6) but are second-order polynomials in the linear momentum \vec{p} .

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Appendix

In this appendix, we give the full symmetric form of the integral of motions separately for scalars, pseudoscalars, vectors, axial vectors, tensors and pseudotensors.

(i) *Scalars*. The full symmetric form of X_S is given as

$$X_S = f_1 + f_2(\vec{x}, \vec{p}) + f_3(\vec{\sigma}, \vec{L}) - \frac{i}{2} r f_2' - \frac{3i}{2} f_2. \quad (\text{A.1})$$

(ii) *Pseudoscalars*. The full symmetric form of X_P could be given as

$$X_P = f_1(\vec{\sigma}, \vec{x}) + f_2(\vec{\sigma}, \vec{p}) + f_3(\vec{\sigma}, \vec{x})(\vec{x}, \vec{p}) - i \frac{(\vec{\sigma}, \vec{x})}{2r} (f_2' + r^2 f_3' + 4r f_3). \quad (\text{A.2})$$

(iii) *Vectors.* The full symmetric form of \vec{X}_V can be written as

$$\begin{aligned} \vec{X}_V = \vec{x} \left(f_1 - \frac{i}{2} \left(\frac{f'_2}{r} + r f'_3 + 4 f_3 \right) + f_3(\vec{x}, \vec{p}) + f_4(\vec{\sigma}, \vec{L}) \right) + f_2 \vec{p} + f_6(\vec{\sigma} \wedge \vec{p}) \\ + f_8(\vec{\sigma}, \vec{x}) \vec{L} - \frac{i}{2} (\vec{\sigma} \wedge \vec{x}) \left(\frac{f'_6}{r} + f_4 - f_8 + r f'_5 + 4 f_5 + 2i(f_5(\vec{x}, \vec{p}) + f_7) \right). \end{aligned} \quad (\text{A.3})$$

(iv) *Axial vectors.* Let us now give the full symmetric form of \vec{X}_A

$$\begin{aligned} \vec{X}_A = f_1 \vec{L} + \vec{\sigma} \left(f_2 - \frac{i}{2} (3 f_3 + r f'_3 + f_4 + f_6) + f_3(\vec{x}, \vec{p}) \right) \\ + \vec{x}(\vec{\sigma}, \vec{x}) \left(f_5 - \frac{i}{2r} (f'_4 + f'_6) - \frac{i}{2} (5 f_7 + r f'_7) + f_7(\vec{x}, \vec{p}) \right) \\ + f_4 \vec{x}(\vec{\sigma}, \vec{p}) + f_6(\vec{\sigma}, \vec{x}) \vec{p}. \end{aligned} \quad (\text{A.4})$$

(v) *Tensors.* In the commutator relation $[H, X_T^{ik}] = 0$, it is enough to consider only one component since the others then necessarily commute due to the rotations. Let us now give the full symmetric form of X_T^{12} :

$$\begin{aligned} X_T^{12} = xy \left(f_1 + f_2(\vec{x}, \vec{p}) - \frac{i}{2} (r f'_2 + 5 f_2) + f_3(\vec{\sigma}, \vec{L}) - i \frac{f'_4}{r} \right) \\ + (zx\sigma_1 - zy\sigma_2 - (x^2 - y^2)\sigma_3) \\ \times \left(\frac{i}{2} \left(f_3 - f_5 + r f'_7 + 5 f_7 + \frac{1}{r} (f'_8 + f'_9) \right) - f_6 - f_7(\vec{x}, \vec{p}) \right) + f_4(xp_y + yp_x) \\ + f_5(\vec{\sigma}, \vec{x})(yL_1 + xL_2) + f_8(y\sigma_2 p_z - y\sigma_3 p_y + x\sigma_3 p_x - x\sigma_1 p_z) \\ - f_9((y\sigma_3 - z\sigma_2)p_y + (z\sigma_1 - x\sigma_3)p_x) + f_{10}(\sigma_2 L_1 + \sigma_1 L_2). \end{aligned} \quad (\text{A.5})$$

(vi) *Pseudotensors.* Similarly, it is enough to consider only one component. The full symmetric form of X_Y^{12} is

$$\begin{aligned} X_Y^{12} = f_1 xy(\vec{\sigma}, \vec{p}) - xy(\vec{\sigma}, \vec{x}) \left(i \frac{f'_1}{2r} - f_2 - f_3(\vec{x}, \vec{p}) + i \frac{r f'_3}{2} + 3i f_3 + i \frac{f'_4}{r} \right) \\ - (y\sigma_1 + x\sigma_2) \left(\frac{i}{2} (f_1 + f_4 + r f'_7 - f_9) - f_6 - f_7(\vec{x}, \vec{p}) + 2i f_7 + i \frac{f'_8}{2r} \right) \\ + f_4(\vec{\sigma}, \vec{x})(xp_y + yp_x) + f_5(yL_1 + xL_2) + f_8(\sigma_1 p_y + \sigma_2 p_x) \\ + f_9((x\sigma_3 - z\sigma_1)L_1 + (z\sigma_2 - y\sigma_3)L_2). \end{aligned} \quad (\text{A.6})$$

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